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# CERTAIN DIFFERENTIAL IDENTITIES INVOLVING SKEW LIE PRODUCT AND GENERALIZED DERIVATIONS

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Abstract. Let $\mathfrak{R}$ be a ring with involution $\eta$. Notation $\nabla[\ell_1, \ell_2]$ is called skew-Lie product and defined by $\ell_1\ell_2 - \ell_2\eta(\ell_1)$. The main objective of this paper is to investigate commutativity of $\eta$-prime rings with involution $\eta$ of the second kind equipped with skew-Lie product involving a generalized derivation. Finally, we furnish some examples which illustrate that the requirements presumed in our results are not redundant.

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## 1. INTRODUCTION

Throughout the paper, $\mathfrak{R}$ will be used to describe an associative ring, and $\vartheta_Z$ is the centre of $\mathfrak{R}$. For any $\ell_1, \ell_2 \in \mathfrak{R}$, symbols $[\ell_1, \ell_2] = \ell_1\ell_2 - \ell_2\ell_1$ is called Lie product (resp. commutator) and $\ell_1 \circ \ell_2 = \ell_1\ell_2 + \ell_2\ell_1$ is called Jordan product (resp. anti-commutator). $\mathfrak{R}$ is called 2-torsion free if $2\ell_1 = 0$ implies $\ell_1 = 0$ for all $\ell_1 \in \mathfrak{R}$. Recalling the definition of an involution on a ring $\mathfrak{R}$. An additive mapping $\eta$ on a ring is called involution if $\eta(ab) = \eta(b)\eta(a)$ and $\eta^2(a) = a$, for all $a, b \in \mathfrak{R}$. A ring $R$ is said to be prime if $aRb = (0)$ (where $a, b \in R$) implies either $a = 0$ or $b = 0$. Ultimately, an involution is an anti-automorphism of order 1 or 2, a ring with involution $\eta$ is called a $\eta$-ring. Prime rings with involution $\eta$ is called $\eta$-prime if $a\mathfrak{R}b = a\mathfrak{R}\eta(b) = (0)$ or $\eta(a)\mathfrak{R}b = a\mathfrak{R}b = (0)$ implies $a = 0, b = 0 \ \forall \ a, b \in \mathfrak{R}$. Every prime ring with involution $\eta$ is a $\eta$-prime ring but the converse is not true in general; for example, let $\mathfrak{R}$ be a prime ring and $S = \mathfrak{R} \times \mathfrak{R}^o$, where $\mathfrak{R}^o$ is an opposite ring of $\mathfrak{R}$. The mapping $\eta$ on $S$ as $\eta(\ell_1, \ell_2) = (\ell_2, \ell_1)$. Then it is easy to check that $S$ with involution $\eta$ is $\eta$-prime ring but $S$ is not a prime ring. We describe an element $\ell_1$ in $\mathfrak{R}$ is said to be hermitian if $\eta(\ell_1) = \ell_1$ and skew-hermitian if $\eta(\ell_1) = -\ell_1$. Let $\vartheta_H$ be the set of hermitian elements and $\vartheta_S$ is a set of skew-hermitian elements of $\mathfrak{R}$. Let $\mathfrak{R}$ be a ring with char$(\mathfrak{R}) \neq 2$, we have for every element $\ell_1 \in \mathfrak{R}$ can be uniquely expressed as $2\ell_1 = h + k$ where $h \in \vartheta_H$ and $k \in \vartheta_S$. An involution $\eta$ is called the first kind if $\vartheta_Z \subseteq \vartheta_H$, otherwise, it is of the second kind. The second kind implies $\vartheta_S \cap \vartheta_Z \neq (0)$ and $\vartheta_H \cap \vartheta_Z \neq (0)$.

A mapping $\psi$ on $\mathfrak{R}$ is called a derivation if $\psi(\ell_1 + \ell_2) = \psi(\ell_1) + \psi(\ell_2)$ and $\psi(\ell_1\ell_2) = \psi(\ell_1)\ell_2 + \ell_1\psi(\ell_2)$ for all $\ell_1, \ell_2 \in \mathfrak{R}$, for any fixed element

$b \in \mathfrak{R}$, a mapping $\psi$ on $\mathfrak{R}$ defined by $\psi(\ell_1) = [b, \ell_1] = b\ell_1 - \ell_1 b$ for all $\ell_1 \in \mathfrak{R}$ is called a inner derivation induced by $b$. An additive mapping $D : \mathfrak{R} \to \mathfrak{R}$ is called a generalized derivation on $\mathfrak{R}$ if there exists a derivation $\psi$ on $\mathfrak{R}$ such that $D(\ell_1\ell_2) = D(\ell_1)\ell_2 + \ell_1\psi(\ell_2)$ for all $\ell_1, \ell_2 \in \mathfrak{R}$. A map $f : \mathfrak{R} \to \mathfrak{R}$ is called centralizing on $\mathfrak{R}$ if $[f(\ell_1), \ell_1] \in \vartheta_Z$ holds for all $\ell_1 \in \mathfrak{R}$. In particular, if $[f(\ell_1), \ell_1] = 0$ holds for all $\ell_1 \in \mathfrak{R}$, then it is called commuting. The history of centralizing and commuting maps began in 1955, when Divinsky established that a simple Artinian ring is commutative if it has a commuting non-trivial automorphisms. Motivated by the representation of a centralizing map, a map $f$ from $\mathfrak{R}$ into itself is called $\eta$-centralizing if $[f(\ell_1), \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$ and is called $\eta$-commuting if $[f(\ell_1), \eta(\ell_1)] = 0$ for all $\ell_1 \in \mathfrak{R}$. Several years later, Posner [17], the presence of a nonzero centralizing derivation on a prime ring guarantees ring commutativity. The study of centralizing (resp. commuting) derivations and various generalizations of the idea of a centralizing (resp. commuting) map are the key topics that emerge immediately from Posner's result, with numerous applications in diverse fields. The commutativity theorem for prime and semi-prime rings with or without involution was recently proved by a number of algebraists, who accepted identities on automorphism, derivations, left centralizers, and generalized derivations, for example, see [2, 4, 7, 11, 12, 13].

In 2016, Ali et al. [2] examine a $\eta$-centralizing derivation in prime rings with involution and showed the $\eta$-version of standard results of Posner [17], and they proved that "If $\mathfrak{R}$ be a prime ring with involution $\eta$ such that char$(\mathfrak{R}) \neq 2$. If $\psi$ is a nonzero derivation of $\mathfrak{R}$ such that $[\psi(\ell_1), \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$ and $\psi(\vartheta_S \cap \vartheta_Z) \neq \{0\}$, then $\mathfrak{R}$ is commutative". Further, this result was extended by Nejjar et al. [14] for the second kind involution instead of condition $\psi(\vartheta_S \cap \vartheta_Z) \neq \{0\}$. Recently, Alahmadi et al. [4] generalized the above result for generalized derivation and they proved that "If $\mathfrak{R}$ is a prime ring with involution $\eta$ of the second kind such that char$(\mathfrak{R}) \neq 2$ and if $\mathfrak{R}$ admits a nonzero generalized derivation $F$ associated with a derivation $d$ such that $[F(t), \eta(t)] \in \vartheta_Z$ for all $t \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative". In this direction a lots of work have been done in the recent years (see for reference [5, 6, 18] where further references can be found).

Our paper's the main goal is to look into a generalized derivations involving skew Lie product on $\eta$-prime rings with involution. Further, we identify the structure of $\eta$-prime rings that satisfy some identities. In fact our results are generalization of some results proved in [3] where authors proved their main result as: "If $\mathfrak{R}$ is a 2-torsion free prime ring with involution $*$ of second kind and admits a generalized derivation $(\mathfrak{F}, d)$ such that $\nabla[x, \mathfrak{F}(x^*)] \pm \nabla[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathfrak{F} = \pm I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping on $\mathfrak{R}$". At the last we provide some examples to demonstrate that the conditions assumed in our results are not unnecessary.

## 2. MAIN RESULTS

**Lemma 2.1.** *Let $\mathfrak{R}$ be a $\eta$-prime ring with involution $\eta$. If $az \in \vartheta_Z$ and $a\eta(z) \in \vartheta_Z$ ehere $a \in \mathfrak{R}$ and $z \in \vartheta_Z$, then $a \in \vartheta_Z$ or $z = 0$.*

*Proof.* Since, $az \in \vartheta_Z$ and $a\eta(z) \in \vartheta_Z$, $0 = [az, r] = [a\eta(z), r]$ for all $r \in \mathfrak{R}$, implies $0 = z[a, r] = \eta(z)[a, r]$. Further implies $(0) = z\mathfrak{R}[a, r] = \eta(z)\mathfrak{R}[a, r]$, by the definition of $\eta$-prime ring, we have either $z = 0$ or $a \in \vartheta_Z$. □

**Lemma 2.2.** *Let $\mathfrak{R}$ be a $\eta$-prime ring with involution $\eta$. If $az \in \vartheta_Z$ and $\eta(a)z \in \vartheta_Z$ for any $a \in \mathfrak{R}$ and $z \in \vartheta_Z$, then $a \in \vartheta_Z$ or $z = 0$.*

*Proof.* Since, $az \in \vartheta_Z$ and $\eta(a)z \in \vartheta_Z$, $0 = [az, r] = [\eta(a)z, r]$ for all $r \in \mathfrak{R}$, implies $0 = z[a, r] = z[\eta(a), r]$ implies $(0) = z\mathfrak{R}[a, r] = z\mathfrak{R}[\eta(a), r]$. Further implies $(0) = z\mathfrak{R}[a, r] = z\mathfrak{R}\eta([a, r])$, by the definition of $\eta$-prime ring, we have either $z = 0$ or $a \in \vartheta_Z$. □

**Lemma 2.3.** [9, Lemma 2.3], *Let $\mathfrak{R}$ be a $\eta$-prime ring of char$(\mathfrak{R}) \neq 2$. Then $\mathfrak{R}$ is 2-torsion free.*

Although it is commonly known that the zero-divisor cannot exist in the centre of a prime ring, but the center of $\eta$-prime rings is not devoid of the zero-divisor. The following example demonstrates the aforementioned fact.

**Example 2.4.** *Consider $\mathfrak{R} = \left\{ \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \Big| \alpha_1, \alpha_2 \in \mathbb{Z} \right\}$, define $\eta$ in such a way $\eta\left( \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \right) = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}$. It is easy to verify that $\mathfrak{R}$ is $\eta$-prime ring with involution $\eta$. For any non-zero $\alpha_1$, $\begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} \in \vartheta_Z$, and for any non-zero $\alpha_2$, $\begin{bmatrix} 0 & 0 \\ 0 & \alpha_2 \end{bmatrix} \in \mathfrak{R}$ and $\begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.*

**Lemma 2.5.** [9, Lemma 2.4], *In $\eta$-prime ring, $\vartheta_Z \cap \vartheta_H$ and $\vartheta_Z \cap \vartheta_S$ are free from zero-divisors.*

**Lemma 2.6.** *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind. Let $\psi$ be a derivation on $\mathfrak{R}$. If $\psi(h) = 0$ for all $h \in \vartheta_H \cap \vartheta_Z$, then $\psi(z) = 0$ for all $z \in \vartheta_Z$.*

*Proof.* By our hypothesis, we have $\psi(h) = 0$, where $h \in \vartheta_H \cap \vartheta_Z$, then $\psi(k^2) = 0$ for $k \in \vartheta_S \cap \vartheta_Z$ implies $k\ \psi(k) = 0$ by Lemma 2.5 we have either $k = 0$ or $\psi(k) = 0$; the first case is not possible because $\eta$ is of the second kind involution. So, we have $\psi(k) = 0$ for $k \in \vartheta_S \cap \vartheta_Z$. For all $z \in \vartheta_Z$ we have for 2-torsion free rings $2z = h + k$. Finally, we have $\psi(2z) = \psi(h) + \psi(k) = 0$ implies $\psi(z) = 0$ for all $z \in \vartheta_Z$. □

**Fact 2.7.** *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind. If $\nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

*Proof.* By the given condition

(1) $$\nabla\ [\ell_1, \eta(\ell_1)] \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}.$$

Linearizing the above equation, we have

(2) $$\nabla\ [\ell_1, \eta(\ell_2)] + \nabla\ [\ell_2, \eta(\ell_1)] \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}.$$

Taking $\ell_2 k$ in place of $\ell_2$, where $k \in \vartheta_Z \cap \vartheta_S$ in the above relation and using Lemma 2.5, we obtain

(3) $$(-\nabla\ [\ell_1, \eta(\ell_2)] + l_2\eta(\ell_1) + \eta(\ell_1)\eta(\ell_2))\ k\ \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}.$$

The last relation further implies

(4) $$[-\nabla\ [\ell_1, \eta(\ell_2)] + l_2\eta(\ell_1) + \eta(\ell_1)\eta(\ell_2),\ r]\ k = 0 \text{ for all } \ell_1, \ell_2, r \in \mathfrak{R}.$$

By Lemma 2.5, we get $k = 0$ or $[-\nabla\ [\ell_1, \eta(\ell_2)] + l_2\eta(\ell_1) + \eta(\ell_1)\eta(\ell_2), r] = 0$ for all $\ell_1, \ell_2, r \in \mathfrak{R}$. The first case is not possible because $\eta$ is of the second kind involution. The later case implies

(5) $$(-\nabla\ [\ell_1, \eta(\ell_2)] + l_2\eta(\ell_1) + \eta(\ell_1)\eta(\ell_2))\ \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}.$$

By using (2) and (5), we have

(6) $$2\ell_2\eta(\ell_1)\ \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}.$$

Since $\mathfrak{R}$ is 2-torsion free, we get

(7) $$\ell_2\eta(\ell_1) \in \vartheta_Z.$$

Replacing $\eta(\ell_1)$ by $\ell_1$ and $\ell_2$ by $k$, where $0 \neq k \in \vartheta_Z \cap \vartheta_S$, we have

(8) $$\ell_1 k \in \vartheta_Z \text{ and } \ell_1\eta(k) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}.$$

By Lemma 2.1, we have

(9) $$\ell_1 \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}.$$

The last relation implies commutativity of $\mathfrak{R}$. □

**Fact 2.8.** [9, Fact 2.2], *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime rings with involution $\eta$ which is of the second kind. If $\eta$ is centralizing, then $\mathfrak{R}$ is commutative.*

**Fact 2.9.** [9, Fact 2.3], *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind. If $\ell_1 \circ \eta(\ell_1) \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

**Fact 2.10.** *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$. If $D(\ell_1) \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then either $\mathfrak{R}$ is commutative or $D = 0$.*

*Proof.* By the given condition

(10) $$D(\ell_1) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}.$$

Let us consider $\vartheta_Z \neq 0$. Taking $\ell_2\ell_1$ in place of $\ell_1$

(11) $$D(\ell_2\ell_1) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}.$$

Commutes the above relation with $\ell_1$, we obtain

(12) $$[\ell_2\psi(\ell_1), \ell_1] = 0 \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}.$$

Replacing $\ell_2$ by $(0 \neq z) \in \vartheta_Z$, we obtain

(13) $$z[\psi(\ell_1), \ell_1] = 0 \text{ for all } \ell_1 \in \mathfrak{R}.$$

From the last relation, we have

(14) $$z\mathfrak{R}[\psi(\ell_1), \ell_1] = (0) = \eta(z)\mathfrak{R}[\psi(\ell_1), \ell_1] \text{ for all } \ell_1 \in \mathfrak{R}.$$

By the definition of $\eta$-prime ring, we have either $z = 0$ or $[\psi(\ell_1), \ell_1] = 0$, the first case is not possible by our assumption, the later case implies

$$[\psi(\ell_1), \ell_1] = 0 \text{ for all } \ell_1 \in \mathfrak{R}. \tag{15}$$

By [15, Theorem 1], $\mathfrak{R}$ is commutative or $\psi = 0$. Replacing $\ell_1$ by $\ell_1 u$, where $u \in \mathfrak{R}$ in (10) and using $\psi = 0$, we obtain

$$D(\ell_1)u \in \vartheta_Z \text{ for all } \ell_1, u \in \mathfrak{R}. \tag{16}$$

The last relation further gives

$$D(\ell_1)\eta(u) \in \vartheta_Z \text{ for all } \ell_1, u \in \mathfrak{R}. \tag{17}$$

The last relation together with (16) and using Lemma 2.2, we obtain

$$\text{either } D(\ell_1) = 0 \text{ for all } \ell_1 \in \mathfrak{R}, \text{ or } u \in \vartheta_Z \text{ for all } u \in \mathfrak{R}. \tag{18}$$

The later case implies commutativity of $\mathfrak{R}$ and the first case implies $D = 0$. □

**Theorem 2.11.** *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$ satisfying $\nabla[\ell_1, D(\eta(\ell_1))] \pm \nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$ then $\mathfrak{R}$ is commutative or $D = \pm I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping on $\mathfrak{R}$.*

*Proof.* Given that

$$\nabla[\ell_1, D(\eta(\ell_1))] + \nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{19}$$

If $D = 0$, then $\nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, so by Fact 2.7, $\mathfrak{R}$ is commutative. Now, for the case $D \neq 0$, replacing $\ell_1$ by $\ell_1 + \ell_2$ in (19), we have

$$\nabla[\ell_1, D(\eta(\ell_2))] + \nabla[\ell_2, D(\eta(\ell_1))] + \nabla[\ell_1, \eta(\ell_2)] + \nabla[\ell_2, \eta(\ell_1)] \in \vartheta_Z. \tag{20}$$

For all $\ell_1, \ell_2 \in \mathfrak{R}$, the last relation further implies

$$\begin{aligned} &\ell_1 D(\eta(\ell_2)) - D(\eta(\ell_2)\eta(\ell_1)) + \ell_2 D(\eta(\ell_1)) - D(\eta(\ell_1)\eta(\ell_2)) + \ell_1\eta(\ell_2) \\ &- \eta(\ell_2)\eta(\ell_1) + \ell_2\eta(\ell_1) - \eta(\ell_1)\eta(\ell_2) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \end{aligned} \tag{21}$$

Replacing $\ell_1$ by $\ell_1 h$, where $h \in \vartheta_H \cap \vartheta_Z$ in the above relation and using it, we obtain

$$(\ell_2\eta(\ell_1) - \eta(\ell_1)\eta(\ell_2))\psi(h) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{22}$$

Replacing $\ell_1$ by $\eta(\ell_1)$ in the last relation, we get

$$\eta(\ell_2\eta(\ell_1) - \eta(\ell_1)\eta(\ell_2))\psi(h) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{23}$$

Combining (22), (23) and using Lemma 2.1, we obtain $(\ell_2\eta(\ell_1) - \eta(\ell_1)\eta(\ell_2)) \in \vartheta_Z$ for all $\ell_1, \ell_2 \in \mathfrak{R}$ or $\psi(h) = 0$ for all $h \in \vartheta_H \cap \vartheta_Z$. The first case implies commutativity of $\mathfrak{R}$ by Fact 2.7. The later case by Lemma 2.6 implies $\psi(z) = 0$ for all $z \in \vartheta_Z$. Replacing $\ell_1$ by $\ell_1 s$ in (21), where $s \in \vartheta_S \cap \vartheta_Z$, we obtain and using $\psi(z) = 0$ for all $z \in \vartheta_Z$

$$\begin{aligned} &(\ell_1 D(\eta(\ell_2)) + D(\eta(\ell_2))\eta(\ell_1) - \ell_2 D(\eta(\ell_1)) + D(\eta(\ell_1))\ell_2 + \ell_1\eta(\ell_2) + \\ &\eta(\ell_2)\eta(\ell_1) - \ell_2\eta(\ell_1) + \eta(\ell_1)\eta(\ell_2))s \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \end{aligned} \tag{24}$$

By using Lemma 2.5, we obtain

$$(\ell_1 D(\eta(\ell_2)) + D(\eta(\ell_2))\eta(\ell_1) - \ell_2 D(\eta(\ell_1)) + D(\eta(\ell_1))\ell_2 + \ell_1\eta(\ell_2)+$$
$$\eta(\ell_2)\eta(\ell_1) - \ell_2\eta(\ell_1) + \eta(\ell_1)\eta(\ell_2)) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{25}$$

The last relation together with (21), we obtain

$$\ell_1 D(\eta(\ell_2)) + \ell_1\eta(\ell_2) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{26}$$

Replacing $\ell_1$ by $h$ and $\ell_2$ by $\eta(\ell_2)$ where $h \in \vartheta_H \cap \vartheta_Z$ and using Lemma 2.5, we obtain

$$D(\ell_2) + \ell_2 \in \vartheta_Z \text{ for all } \ell_2 \in \mathfrak{R}. \tag{27}$$

The last relation further gives

$$(D + I_{\mathfrak{R}})(\ell_2) \in \vartheta_Z \text{ for all } \ell_2 \in \mathfrak{R}. \tag{28}$$

Here, $I_{\mathfrak{R}}$ represent the identity mapping on $\mathfrak{R}$ and $D + I_{\mathfrak{R}}$ is a generalized derivation associated with a derivation $\psi$, by Fact 2.10, we have either $\mathfrak{R}$ is commutative or $D = -I_{\mathfrak{R}}$. When we take $\nabla[\ell_1, D(\eta(\ell_1))] - \nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then by same process we obtain the required result. □

**Corollary 2.12.** [3, Theorem 4], *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$ satisfying $\nabla[\ell_1, D(\eta(\ell_1))] \pm \nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$ then $\mathfrak{R}$ is commutative or $D = \pm I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping on $\mathfrak{R}$.*

**Theorem 2.13.** *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$ satisfying $\nabla[\ell_1, D(\ell_1)] + \nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

*Proof.* Given that

$$\nabla[\ell_1, D(\ell_1)] + \nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{29}$$

The last relation further implies that

$$\ell_1 D(\ell_1) - D(\ell_1)\eta(\ell_1) + \ell_1\eta(\ell_1) - \eta(\ell_1)^2 \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{30}$$

Replacing $\ell_1$ by $\ell_1 h$ in the above relation and using it, where $h \in \vartheta_H \cap \vartheta_Z$, we obtain

$$(\ell_1^2 - \ell_1\eta(\ell_1))\psi(h) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{31}$$

Replacing $\ell_1$ by $\ell_1 s$ in the above relation, where $s \in \vartheta_S \cap \vartheta_Z$ and using Lemma 2.1, we obtain

$$(\ell_1^2 + \ell_1\eta(\ell_1))\psi(h) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{32}$$

Combining (32) and (31), we have

$$\ell_1\eta(\ell_1)\psi(h) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{33}$$

The last relation further implies

$$\eta(\ell_1\eta(\ell_1))\psi(h) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{34}$$

The last relation together with (33) and by Lemma 2.2, we obtain either $\ell_1\eta(\ell_1) \in \vartheta_Z$ or $\psi(h) = 0$, the first case further implies $\ell_1 \circ \eta(\ell_1) \in \vartheta_Z$, so by Fact 2.9, $\mathfrak{R}$ is commutative. The later case implies $\psi(z) = 0$ for all $z \in \vartheta_Z$.

Replacing $\ell_1$ by $\ell_1 s$ in (30), where $s \in \vartheta_S \cap \vartheta_Z$ and using Lemma 2.1, we obtain

$$\ell_1 D(\ell_1) + D(\ell_1)\eta(\ell_1) - \ell_1\eta(\ell_1) - \eta(\ell_1)^2 \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{35}$$

Combining (30) and (35), we obtain

$$\ell_1 D(\ell_1) - \eta(\ell_1)^2 \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{36}$$

Replacing $\ell_1$ by $\ell_1 + \ell_2$, in the above equation we obtain

$$\ell_1 D(\ell_2) + \ell_2 D(\ell_1) - \eta(\ell_1)\eta(\ell_2) - \eta(\ell_2)\eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{37}$$

Replacing $\ell_1$ by $\ell_1 s$, in the above equation, where $0 \neq s \in \vartheta_Z \cap \vartheta_S$ and using $\psi(z) = 0$ for all $z \in \vartheta_Z$, we obtain

$$(\ell_1 D(\ell_2) + \ell_2 D(\ell_1) + \eta(\ell_1)\eta(\ell_2) + \eta(\ell_2)\eta(\ell_1))s \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{38}$$

By Lemma 2.5, we obtain

$$\ell_1 D(\ell_2) + \ell_2 D(\ell_1) + \eta(\ell_1)\eta(\ell_2) + \eta(\ell_2)\eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{39}$$

Combining (39) and (37), we obtain

$$\eta(\ell_1)\eta(\ell_2) + \eta(\ell_2)\eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{40}$$

In particular, we obtain $\ell_1 \circ \eta(\ell_1) \in \vartheta_Z$ for all $\ell_1 \in \vartheta_Z$, so by Fact 2.9, $\mathfrak{R}$ is commutative. □

**Corollary 2.14.** [3, Theorem 5]*, Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$ satisfying $\nabla[\ell_1, D(\ell_1)] + \nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

**Theorem 2.15.** *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$ satisfying $\nabla[\ell_1, D(\ell_1)] + [\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

*Proof.* Given that

$$\nabla[\ell_1, D(\ell_1)] + [\ell_1, \eta(\ell_1)] \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{41}$$

Replacing $\ell_1$ by $\ell_1 + \ell_2$, in the above relation, we obtain

$$\nabla[\ell_1, D(\ell_2)] + \nabla[\ell_2, D(\ell_1)] + [\ell_1, \eta(\ell_2)] + [\ell_2, \eta(\ell_1)] \in \vartheta_Z. \tag{42}$$

For all $\ell_1, \ell_2 \in \mathfrak{R}$, the last relation further implies

$$\begin{aligned} &\ell_1 D(\ell_2) - D(\ell_2)\eta(\ell_1) + \ell_2 D(\ell_1) - D(\ell_1)\eta(\ell_2) + \ell_1\eta(\ell_2) \\ &+ \ell_2\eta(\ell_1) - \eta(\ell_1)\ell_2 - \eta(\ell_2)\ell_1 \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \end{aligned} \tag{43}$$

Replacing $\ell_1$ by $\ell_1 h$ in (43) and using it, where $0 \neq h \in \vartheta_H \cap \vartheta_Z$, we obtain

$$(\ell_2\ell_1 - \ell_1\eta(\ell_2))\psi(h) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{44}$$

Replacing $\ell_1$ by $\eta(\ell_1)$, in Equation (44), we get

$$\eta(\ell_2\ell_1 - \ell_1\eta(\ell_2))\psi(h) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{45}$$

Combining (44) and (45) and then using Lemma 2.2, we obtain either $\ell_2\ell_1 - \ell_1\eta(\ell_2) \in \vartheta_Z$ or $\psi(h) = 0$ for all $0 \neq h \in \vartheta_H \cap \vartheta_Z$. The first case implies $\nabla[\ell_1, \eta(\ell_1)] \in \vartheta_Z$, so by Fact 2.7, $\mathfrak{R}$ is commutative, the later case implies

$\psi(z) = 0$ for all $z \in \vartheta_Z$, replacing $\ell_1$ by $\ell_1 s$ in (43), where $0 \neq s \in \vartheta_S \cap \vartheta_Z$ and using Lemma 2.5, we obtain

$$\begin{aligned} &\ell_1 D(\ell_2) + D(\ell_2)\eta(\ell_1) + \ell_2 D(\ell_1) - D(\ell_1)\eta(\ell_2) + \ell_1\eta(\ell_2) \\ &- \ell_2\eta(\ell_1) + \eta(\ell_1)\ell_2 - \eta(\ell_2)\ell_1 \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \end{aligned} \tag{46}$$

Combining (43) and (46), we obtain

$$D(\ell_2)\eta(\ell_1) - \ell_2\eta(\ell_1) + \eta(\ell_1)\ell_2 \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{47}$$

Replacing $\ell_1$ by $h$ in the last equation where $0 \neq h \in \vartheta_H \cap \vartheta_Z$, we obtain

$$D(\ell_2)h \in \vartheta_Z \text{ for all } \ell_2 \in \mathfrak{R}. \tag{48}$$

By Lemma 2.5, we obtain

$$D(\ell_2) \in \vartheta_Z \text{ for all } \ell_2 \in \mathfrak{R}. \tag{49}$$

By Fact 2.10, we have either $\mathfrak{R}$ is commutative or $D = 0$, the later case from (41), we obtain $[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, so by Fact 2.8, $\mathfrak{R}$ is commutative. □

**Corollary 2.16.** [3, Theorem 6], *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$, if $\nabla[\ell_1, D(\ell_1)] + [\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

**Theorem 2.17.** *Let $\mathfrak{R}$ be a 2-torsion free $\eta$-prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$, if $\nabla[\ell_1, D(\eta(\ell_1))] \pm \ell_1 \circ \eta(\ell_1) \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

*Proof.* Given that

$$\nabla[\ell_1, D(\eta(\ell_1))] + \ell_1 \circ \eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{50}$$

If $D = 0$, then $\mathfrak{R}$ is commutative by Fact 2.9. If $D \neq 0$, then by linearization of the last relation implies

$$\begin{aligned} &\nabla[\ell_1, D(\eta(\ell_2))] + \nabla[\ell_2, D(\eta(\ell_1))] + \ell_1 \circ \eta(\ell_2) \\ &+ \ell_2 \circ \eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \end{aligned} \tag{51}$$

The last relation further implies

$$\begin{aligned} &\ell_1 D(\eta(\ell_2)) - D(\eta(\ell_2))\eta(\ell_1) + \ell_2 D(\eta(\ell_1)) - D(\eta(\ell_1))\eta(\ell_2) \\ &+ \ell_1 \circ \eta(\ell_2) + \ell_2 \circ \eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \end{aligned} \tag{52}$$

Replacing $\ell_1$ by $\ell_1 h$ in (52) and using it, where $0 \neq h \in \vartheta_H \cap \vartheta_Z$, we obtain

$$\{\ell_2\eta(\ell_1) - \eta(\ell_1)\eta(\ell_2)\}\psi(h) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{53}$$

Now, the above relation is same as in (22), we get $\mathfrak{R}$ is commutative or $\psi(z) = 0$ for all $z \in \vartheta_Z$. The later case implies, replacing $\ell_1$ by $\ell_1 s$ in (52), where $0 \neq s \in \vartheta_S \cap \vartheta_Z$, we obtain

$$\begin{aligned} &\{\ell_1 D(\eta(\ell_2)) + D(\eta(\ell_2))\eta(\ell_1) - \ell_2 D(\eta(\ell_1)) + D(\eta(\ell_1))\eta(\ell_2) \\ &+ \ell_1 \circ \eta(\ell_2) - \ell_2 \circ \eta(\ell_1)\}s \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \end{aligned} \tag{54}$$

By using Lemma 2.5, we obtain

$$\ell_1 D(\eta(\ell_2)) + D(\eta(\ell_2))\eta(\ell_1) - \ell_2 D(\eta(\ell_1)) + D(\eta(\ell_1))\eta(\ell_2) + \ell_1 \circ \eta(\ell_2) - \ell_2 \circ \eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{55}$$

The last relation together with (52), implies

$$\ell_1 D(\eta(\ell_2)) + \ell_1 \eta(\ell_2) + \eta(\ell_2)\ell_1 \in \vartheta_Z \text{ for all } \ell_1, \ell_2 \in \mathfrak{R}. \tag{56}$$

Replacing $\ell_1$ by $h$ and $\ell_2$ by $\eta(\ell_2)$ in the above relation where $0 \neq h \in \vartheta_H \cap \vartheta_Z$, we obtain

$$D(\ell_2) + 2\ell_2 \in \vartheta_Z \text{ for all } \ell_2 \in \mathfrak{R}. \tag{57}$$

The last relation further implies

$$(D + 2I_{\mathfrak{R}})(\ell_2) \in \vartheta_Z \text{ for all } \ell_2 \in \mathfrak{R}. \tag{58}$$

By Fact 2.10, we have either $\mathfrak{R}$ is commutative or $D = -2I_{\mathfrak{R}}$ the later case together with (50), we obtain

$$\nabla[\ell_1, -2\eta(\ell_1)] + \ell_1 \circ \eta(\ell_1) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{59}$$

The last relation further implies

$$2(\eta(\ell_1))^2 - [\ell_1, \eta(\ell_1)] \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{60}$$

Replacing $\ell_1$ by $\ell_1 s$ in the above relation where $0 \neq s \in \vartheta_S \cap \vartheta_Z$, we obtain

$$2(\eta(\ell_1)^2 + [\ell_1, \eta(\ell_1)])s^2 \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{61}$$

By Lemma 2.5, we obtain

$$2(\eta(\ell_1)^2 + [\ell_1, \eta(\ell_1)]) \in \vartheta_Z \text{ for all } \ell_1 \in \mathfrak{R}. \tag{62}$$

Combining (62) and (60), we obtain $[\ell_1, \eta(\ell_1)] \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, Fact 2.8, implies commutativity of $\mathfrak{R}$. Now, we have $\nabla[\ell_1, D(\eta(\ell_1))] - \ell_1 \circ \eta(\ell_1) \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, so by the same process $\mathfrak{R}$ is commutative. □

**Corollary 2.18.** [3, Theorem 7], *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $\eta$ which is of the second kind and $D$ be a generalized derivation associated with a derivation $\psi$ on $\mathfrak{R}$, if $\nabla[\ell_1, D(\eta(\ell_1))] \pm \ell_1 \circ \eta(\ell_1) \in \vartheta_Z$ for all $\ell_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

The following example shows that the second kind is necessary in Theorems 2.15 and 2.17.

**Example 2.19.** *Consider $\mathfrak{R} = \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \Big| \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} \right\}$, define $\eta$ in such a way $\eta\left(\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}\right) = \begin{bmatrix} \alpha_4 & -\alpha_2 \\ -\alpha_3 & \alpha_1 \end{bmatrix}$. It is easy to verify that $\mathfrak{R}$ is $\eta$-prime ring with involution $\eta$ which is of the first kind. Moreover, we define a generalized derivation $D$ and a derivation $\psi$ as $D\left(\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}\right) = \begin{bmatrix} 0 & -\alpha_2 \\ \alpha_3 & 0 \end{bmatrix}$ and $\psi = D$, here $D$ is a generalized derivation associated with a derivation $\psi$ satisfy the condition of Theorem* 2.15 *and* 2.17, *however $\mathfrak{R}$ is non-commutative.*

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